

Boundary asymptotic analysis for an incompressible viscous flow: Navier wall laws

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Abstract

We consider a new way of establishing Navier wall laws. Considering a bounded domain Ω of \mathbf{R}^N , $N = 2, 3$, surrounded by a thin layer Σ_ε , along a part Γ_2 of its boundary $\partial\Omega$, we consider a Navier-Stokes flow in $\Omega \cup \partial\Omega \cup \Sigma_\varepsilon$ with Reynolds' number of order $1/\varepsilon$ in Σ_ε . Using Γ -convergence arguments, we describe the asymptotic behaviour of the solution of this problem and get a general Navier law involving a matrix of Borel measures having the same support contained in the interface Γ_2 . We then consider two special cases where we characterize this matrix of measures. As a further application, we consider an optimal control problem within this context.

Navier law, Navier-Stokes flow, Γ -convergence, asymptotic behaviour, optimal control problem.

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1 Introduction

A common hypothesis used in fluid mechanics is that, at the interface between a solid and a fluid, the velocity u of the fluid is equal to that of the solid. If the solid is at rest, the velocity of the fluid must thus vanish: $u = 0$, on the boundary of the solid. These are the so-called rigid boundary conditions. When writing this condition, one assumes that the fluid perfectly adheres to the solid.

This hypothesis has not always been accepted for a viscous fluid, although some verifications have been made through experiments. G. Taylor indeed verified in 1923 the correctness of this hypothesis, when studying the stability of the motion of a fluid flowing between two cylinders in rotation (Taylor-Couette's problem).

Another approach has then been suggested. A thin layer adhering to the solid exists with a tangential velocity different from 0 on the surface of the solid. Navier suggested that this tangential velocity is proportional to the shearing strains and thus is given through

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} &= \kappa u, \\ u \cdot n &= 0, \end{cases}$$

where Id is the identity matrix, n is the unit outer normal vector to the surface of the solid, ν is the viscosity of the fluid and κ is a proportionality coefficient.

Many works have already been devoted to the derivation of Navier boundary conditions, see for example [2], [3], [13] and [14]. In [2] and [3], the authors considered a viscous and incompressible fluid, whose Reynolds number is of order $1/\varepsilon$, flowing in a domain with rugosities of thinness ε and ε -periodically

distributed on its boundary surface, and assuming an homogeneous Dirichlet boundary condition on the boundary of these rugosities. Using the asymptotic expansion method, they deduced, at the first-order level, a kind of Navier wall law

$$\begin{cases} \varepsilon (Id - n \otimes n) \nu \frac{\partial u}{\partial n} &= \kappa u, \\ u \cdot n &= 0. \end{cases}$$

In [13], the authors considered the laminar flow in a pipe with rough pieces ε -periodically distributed on the surface of the pipe, and imposing an homogeneous Dirichlet boundary condition on the boundary of these rough pieces. They used an homogenization process and obtained a Navier wall law, computing a corrector term. In [14], the author considered an ε -periodic geometry built with rough pieces of thinness ε^m and imposed there a boundary condition of the type

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u^\varepsilon}{\partial n} &= \varepsilon^k (g^\varepsilon - \kappa u^\varepsilon), \\ u^\varepsilon \cdot n &= 0. \end{cases}$$

The following limit law was obtained, depending on k and m

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} &= \lambda (g - \kappa u), \\ u \cdot n &= 0. \end{cases}$$

Throughout the present work, we consider a bounded domain $\Omega \subset \mathbf{R}^N$, $N = 2, 3$, whose boundary $\partial\Omega$ is Lipschitz continuous. We suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, with $|\Gamma_1|, |\Gamma_2| > 0$, where $|\Gamma_i|$ denotes the Lebesgue measure of Γ_i . We suppose that near Γ_2 there exists a thin layer Σ_ε of thinness $\varepsilon > 0$, which extends Ω into $\Omega_\varepsilon = \Omega \cup \Gamma_2 \cup \Sigma_\varepsilon$.

Figure 1: The domain under consideration.

We consider the steady-state, viscous and incompressible Navier-Stokes flow in Ω_ε

$$\begin{cases} -\nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon &= f & \text{in } \Omega, \\ -\nu \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon &= f & \text{in } \Sigma_\varepsilon, \\ \operatorname{div}(u^\varepsilon) &= 0 & \text{in } \Omega_\varepsilon, \\ (u^\varepsilon)^+ &= (u^\varepsilon)^- & \text{on } \Gamma_2, \\ \nu \left(\frac{\partial u^\varepsilon}{\partial n} \right)^+ &= \nu \varepsilon \left(\frac{\partial u^\varepsilon}{\partial n} \right)^- & \text{on } \Gamma_2, \\ u^\varepsilon &= 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad (1)$$

where the superscript $+$ (resp. $-$) denotes the trace seen from Ω (resp. from Σ_ε) on Γ_2 . The thin layer Σ_ε is here considered as an unstable thin boundary layer whose Reynolds' number R_ε is of order $1/\varepsilon$ (see [12, pages 239-240], where Reynolds' number is allowed to depend on the thinness of the layer). In the problem (1), we suppose that the density f of volumic forces belongs to $\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)$.

Our purpose is to describe the asymptotic behavior of the solution u^ε of (1) when ε goes to 0, in order to derive the Navier wall law. We use Γ -convergence arguments (see [5] for the definition and the

properties of the Γ -convergence) in order to characterize the limit problem. Our approach is based on the tools developed in [1], [4], [7], [8] and [9]. On Γ_2 , we will get a general Navier law of the kind

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} + \mu^\bullet u &= 0, \\ u \cdot n &= 0, \end{cases}$$

where μ^\bullet is a symmetric matrix $(\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having their support contained in Γ_2 , which do not charge the polar subsets of \mathbf{R}^N and which satisfy $\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^N, \forall B \in \mathcal{B}(\mathbf{R}^N)$, where $\mathcal{B}(\mathbf{R}^N)$ denotes the set of all Borel subsets of \mathbf{R}^N and where we have used the summation convention with respect to repeated indices.

As a first special case, we prove that when $\Omega \subset \{x_3 > 0\}$, $\Gamma_2 = \partial\Omega \cap \{x_3 = 0\}$ and

$$\Sigma_\varepsilon = \left\{ x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon h\left(\frac{x'}{\varepsilon}\right) < x_3 < 0 \right\},$$

where h is a periodic function, we get on Γ_2 the Robin type boundary conditions

$$\begin{cases} \frac{\partial u_1}{\partial x_3}(x', 0) &= -c_1 u_1(x', 0), \\ \frac{\partial u_2}{\partial x_3}(x', 0) &= -c_2 u_2(x', 0), \\ u_3(x', 0) &= 0, \end{cases}$$

where $c_m, m = 1, 2$, are constants which will be computed in terms of the solution of appropriate local thin layer problems (21). This situation can be generalized to the case of a general open and bounded set Ω , surrounded on a part of its boundary by such a rough thin layer.

As a second example, we will consider the case where

$$\Sigma_\varepsilon = \{s + tn(s) \mid s \in \Gamma_2, -\varepsilon h(s) < x_3 < 0\},$$

where h is a Lipschitz continuous and positive function on Γ_2 . We here prove that Navier's law takes the following expression on Γ_2

$$\begin{cases} (Id - n \otimes n) \frac{\partial u}{\partial n} + \frac{1}{h} u &= 0, \\ u \cdot n &= 0. \end{cases}$$

In the last part of this work, we consider an optimal control problem. Choosing $m > 0$, we consider the set Ξ_m of all the matrices $\mathbf{h} = \text{Diag}(h_i)_{i=1,\dots,N}$ of functions $h_i : \Gamma_2 \rightarrow [0, +\infty]$, which are $d\Gamma_2$ -measurable and satisfy $\int_{\Gamma_2} h_i d\Gamma_2 = m, \forall i = 1, \dots, N$. We suppose that Ω is smooth enough and consider the following problem with Navier conditions on Γ_2

$$\begin{cases} -\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h &= f \quad \text{in } \Omega, \\ \text{div}(u^h) &= 0 \quad \text{in } \Omega, \\ \mathbf{h}(Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h &= 0 \quad \text{on } \Gamma_2, \\ u^h \cdot n &= 0 \quad \text{on } \Gamma_2. \end{cases} \quad (2)$$

Let (u^h, p^h) be the solution of (2) and define the functional \mathbf{F} through

$$\mathbf{F}(\mathbf{h}, u) = \begin{cases} \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 \\ \quad + \int_{\Omega} (u^h \cdot \nabla) u^h \cdot v dx - \int_{\Omega} f \cdot u dx & \text{if } u \in \mathbf{V}_{0,\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathbf{V}_{0,\Gamma_1}(\Omega)$ is the functional space defined in (7). We consider the optimal control problem

$$\min_{\mathbf{h} \in \Xi_m} \min_{u \in \mathbf{V}_{0,\Gamma_1}(\Omega)} \mathbf{F}(\mathbf{h}, u). \quad (3)$$

In the last section of this work, we describe the asymptotic behavior of the solution of (3), when m goes to 0, and characterize the zones where some thin boundary layer appears. A problem of this kind has been considered in [11], but for a linear diffusion problem.

2 Functional framework

We define the $(H^1(\mathbf{R}^N))$ capacity of any compact subset K of \mathbf{R}^N as

$$\begin{aligned} \text{Cap}(K) \\ = \inf \left\{ \int_{\mathbf{R}^N} |\nabla \varphi|^2 dx + \int_{\mathbf{R}^N} |\varphi|^2 dx \mid \varphi \in \mathbf{C}_c^\infty(\mathbf{R}^N), \varphi \geq 1 \text{ on } K \right\}. \end{aligned}$$

If U is an open subset of \mathbf{R}^N , then we define

$$\text{Cap}(U) = \sup \{ \text{Cap}(K) \mid K \subset U, K \text{ compact} \}.$$

If $B \subset \mathbf{R}^N$ is a Borel subset of \mathbf{R}^N , then we define

$$\text{Cap}(B) = \inf \{ \text{Cap}(U) \mid B \subset U, U \text{ open} \}.$$

Definition 1 Let $\mathcal{B}(\mathbf{R}^N)$ be the σ -algebra of all Borel subsets of \mathbf{R}^N .

1. A property is said to be true quasi-everywhere (q.e.) on $B \in \mathcal{B}(\mathbf{R}^N)$ if it is true except on a subset of B of capacity Cap equal to 0.
2. A function $u : B \rightarrow \overline{\mathbf{R}}$, with $B \in \mathcal{B}(\mathbf{R}^N)$, is quasi-continuous on B if, for every $\varepsilon > 0$, there exists an open subset $U \subset B$ with $\text{Cap}(U) < \varepsilon$ and such that the restriction of u on $B \setminus U$ is continuous.
3. Every function $u \in \mathbf{H}^1(\mathbf{R}^N)$ has a quasi-continuous representative \tilde{u} , which is unique for the equality quasi-everywhere in \mathbf{R}^N , (see [17], for example). \tilde{u} is given through

$$\tilde{u}(x) = \lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy,$$

for q.e. $x \in \mathbf{R}^N$, where $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$ of \mathbf{R}^N of radius $r > 0$ and centered at x .

We define some notions concerning families of subsets of \mathbf{R}^N .

Definition 2 1. A subset $\mathcal{D} \subset \mathcal{B}(\mathbf{R}^N)$ is a dense family in $\mathcal{B}(\mathbf{R}^N)$ if, for every $A, B \in \mathcal{B}(\mathbf{R}^N)$ with $\overline{A} \subset \overset{\circ}{B}$, there exists $D \in \mathcal{D}$ such that: $\overline{A} \subset \overset{\circ}{D} \subset \overline{D} \subset \overset{\circ}{B}$, where $\overset{\circ}{A}$ (resp. \overline{A}) denotes the interior (resp. the closure) of A .

2. A subset $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^N)$ is a rich family in $\mathcal{B}(\mathbf{R}^N)$ if, for every family $(A_t)_{t \in]0, 1[} \subset \mathcal{B}(\mathbf{R}^N)$ such that $\overline{A}_s \subset \overset{\circ}{A}_t$, for every $s < t$, the set $\{t \in]0, 1[\mid A_t \notin \mathcal{R}\}$ is at most countable.

Let $\mathcal{O}(\mathbf{R}^N)$ be the set of all open subsets of \mathbf{R}^N . We consider the class \mathbb{F} of functionals F from $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$ to $[0, +\infty]$ satisfying:

- i) (*Lower semi-continuity*): for every open subset $\omega \in \mathcal{O}(\mathbf{R}^N)$, the functional $u \mapsto F(u, \omega)$ is lower semi-continuous with respect to the strong topology of $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$;
- ii) (*Measure property*): for every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$, $\omega \mapsto F(u, \omega)$ is the restriction to $\mathcal{O}(\mathbf{R}^N)$ of some Borel measure still denoted $F(u, \omega)$;
- iii) (*Localization*): for every $\omega \in \mathcal{O}(\mathbf{R}^N)$ and every $u, v \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$:

$$u|_\omega = v|_\omega \Rightarrow F(u, \omega) = F(v, \omega);$$

- iv) (\mathbf{C}^1 -convexity): for every $\omega \in \mathcal{O}(\mathbf{R}^N)$, the functional $u \mapsto F(u, \omega)$ is convex on $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and moreover

$$\forall \varphi \in \mathbf{C}^1(\mathbf{R}^N), 0 \leq \varphi \leq 1 : F(\varphi u + (1 - \varphi)v, \omega) \leq F(u, \omega) + F(v, \omega).$$

Example 3 Let us define $\Gamma_{2,\varepsilon} = \partial\Omega_\varepsilon \cap \overline{\Sigma_\varepsilon}$, for some thin layer Σ_ε , as defined above. We consider the functional F^ε defined on the space $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$ through

$$F^\varepsilon(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} = 0, \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

One can prove that F^ε belongs to \mathbb{F} , for every $\varepsilon > 0$.

Let us set the following definitions.

Definition 4 Let Cap be the above-defined capacity.

1. A Borel measure λ is absolutely continuous with respect to the capacity Cap if

$$\forall B \in \mathcal{B}(\mathbf{R}^N) : Cap(B) = 0 \Rightarrow \lambda(B) = 0.$$

2. \mathcal{M}_0 is the set of nonnegative Borel measures \mathbf{R}^N which are absolutely continuous with respect to the capacity Cap .

We have the following example of measure in \mathcal{M}_0 .

Example 5 For every $E \subset \mathbf{R}^N$ such that $Cap(E) > 0$, we define the measure ∞_E through

$$\infty_E(B) = \begin{cases} 0 & \text{if } Cap(B \cap E) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\infty_E \in \mathcal{M}_0$.

Notice that, for every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and every $\omega \in \mathcal{O}(\mathbf{R}^N)$, the functional F^ε defined in (4) can be written as

$$F^\varepsilon(u, \omega) = \int_\omega |\tilde{u}|^2 d\infty_{\Gamma_{2,\varepsilon}} = \int_\omega |u|^2 d\infty_{\Gamma_{2,\varepsilon}}.$$

One has the following representation theorem for the functionals of \mathbb{F} .

Theorem 6 (see [9]) For every $F \in \mathbb{F}$, there exist a finite measure $\lambda \in \mathcal{M}_0$, a nonnegative Borel measure ν and a Borel function $g : \mathbf{R}^N \times \mathbf{R}^N \rightarrow [0, +\infty]$, with $\zeta \mapsto g(x, \zeta)$ convex and lower semi-continuous on \mathbf{R}^N , such that

$$\forall u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N), \forall \omega \in \mathcal{O}(\mathbf{R}^N) : F(u, \omega) = \int_\omega g(x, \tilde{u}(x)) d\lambda + \nu(\omega).$$

Throughout the paper, we will need the following Corollary (see [9, Corollary 8.4]).

Corollary 7 *Let $F \in \mathbb{F}$. If $F(., \omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbf{R}^N)$, there exist $\lambda \in \mathcal{M}_0$ finite, a symmetric matrix $(a_{ij})_{i,j=1,\dots,N}$, of Borel functions from \mathbf{R}^N to \mathbf{R} satisfying $a_{ij}(x)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$ and for q.e. $x \in \mathbf{R}^N$, for every $x \in \mathbf{R}^N$ a subspace $\mathbf{V}(x)$ of \mathbf{R}^N , such that, for every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and every $\omega \in \mathcal{O}(\mathbf{R}^N)$:*

a) *if $F(u, \omega) < +\infty$, then $u(x) \in \mathbf{V}(x)$, for q.e. $x \in \omega$,*

b) *if $u(x) \in \mathbf{V}(x)$, for q.e. $x \in \omega$*

$$F(u, \omega) = \int_{\omega} a_{ij} u_i u_j d\lambda. \quad (5)$$

Remark 8 *Let $F \in \mathbb{F}$, $\lambda \in \mathcal{M}_0$ be the associated measure and Λ be the set defined as $\Lambda = \cup_{\omega \in A(F)} \omega$, where*

$$A(F) = \{\omega \in \mathcal{O}(\mathbf{R}^N) \mid F(., \omega) < +\infty, \text{ for q.e. } x \in \omega\}.$$

We define the matrix $\mu^\bullet = (\mu_{ij}) = (a_{ij}\lambda)_{i,j=1,\dots,N} + \infty_{\mathbf{R}^N \setminus \Lambda} Id$ of measures, and, for every $x \in \mathbf{R}^N$, the subspace $\mathbf{V}(x)$ through

$$\mathbf{V}(x) = \begin{cases} \mathbf{R}^N & \text{if } x \in \Lambda, \\ \{0\} & \text{if } x \in \mathbf{R}^N \setminus \Lambda. \end{cases} \quad (6)$$

For every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and every $\omega \in \mathcal{O}(\mathbf{R}^N)$, one has, using the preceding definition of μ^\bullet

$$\int_{\omega} u_i u_j d\mu_{ij} = \begin{cases} \int_{\omega} a_{ij} u_i u_j d\lambda & \text{if } \omega \subset \Lambda, \\ \int_{\omega \cap \Lambda} a_{ij} u_i u_j d\lambda & \text{if } \begin{cases} u(x) = 0, \forall x \in \omega \cap \mathbf{R}^N \setminus \Lambda \\ \text{and } Cap(\omega \cap \mathbf{R}^N \setminus \Lambda) > 0, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

Thanks to (6), this expression can be written as

$$\int_{\omega} u_i u_j d\mu_{ij} = \begin{cases} \int_{\omega} a_{ij} u_i u_j d\lambda & \text{if } u(x) \in \mathbf{V}(x), \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise.} \end{cases}$$

We can thus write the functional F defined in (5) as

$$F(u, \omega) = \int_{\omega} u_i u_j d\mu_{ij}.$$

3 Study of the problem (1)

We here suppose that the "outer" boundary $\Gamma_{2,\varepsilon}$ of Σ_ε can be defined as

$$\Gamma_{2,\varepsilon} = \{(s, t) \mid s \in \Gamma_2, t = -\varepsilon h_\varepsilon(s)\},$$

where h_ε is a locally Lipschitz continuous function satisfying

$$\|h_\varepsilon\|_{\mathbf{L}^\infty(\Gamma_2)} \leq C, \quad \forall \varepsilon > 0,$$

for some constant C independent of ε . The Lipschitz continuity of h_ε ensures the almost everywhere existence of a unit outer normal vector to $\Gamma_{2,\varepsilon}$, thanks to Rademacher's Theorem, and ensures the

existence of an extension of every function of $\mathbf{H}^1(\Omega_\varepsilon, \mathbf{R}^N)$ in a function of $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$. Let us define the functional spaces

$$\begin{aligned} \mathbf{L}^2(\mathbf{R}^N, \text{div}) &= \left\{ u \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbf{R}^N \right\}, \\ \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div}) &= \left\{ u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbf{R}^N, \right. \\ &\quad \left. u = 0 \text{ on } \Gamma_1 \right\}, \\ \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}) &= \left\{ u \in \mathbf{H}^1(\Omega, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma_1 \right\}, \\ \mathbf{V}_{\Gamma_1}(\Omega) &= \mathbf{L}^2(\mathbf{R}^N, \text{div}) \cap \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}), \\ \mathbf{V}_{0, \Gamma_1}(\Omega) &= \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}) \cap \left\{ u \in \mathbf{H}^1(\Omega, \mathbf{R}^N) \mid u \cdot n = 0 \text{ on } \Gamma_2 \right\}. \end{aligned} \quad (7)$$

In (1), let us replace throughout this section the homogeneous Dirichlet boundary condition $u^\varepsilon = 0$, on $\partial\Omega_\varepsilon$ by a combination between the homogeneous Dirichlet boundary condition $u^\varepsilon = 0$, on $\Gamma_{2, \varepsilon} \cap \omega$, for a given $\omega \in \mathcal{O}(\mathbf{R}^N)$, and homogeneous Neumann boundary conditions on $\Gamma_{2, \varepsilon} \setminus (\Gamma_{2, \varepsilon} \cap \omega)$. We introduce the functional space adapted to (1), with these modified boundary conditions

$$\mathbf{V}_{0, \omega}(\Omega_\varepsilon) = \left\{ v \in \mathbf{H}^1(\Omega_\varepsilon, \mathbf{R}^N) \mid \text{div}(v) = 0 \text{ in } \Omega_\varepsilon, \right. \\ \left. v = 0 \text{ on } \Gamma_1 \cup (\Gamma_{2, \varepsilon} \cap \omega) \right\}.$$

The variational formulation of (1) can be written as

$$\begin{aligned} \forall \varphi \in \mathbf{V}_{0, \omega}(\Omega_\varepsilon) : \nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \varphi dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx \\ + \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \varphi dx = \int_{\Omega_\varepsilon} f \cdot \varphi dx. \end{aligned} \quad (8)$$

Thanks to [15], for example, we deduce that (1) has a unique solution $(u^\varepsilon, p^\varepsilon)$ belonging to the space $\mathbf{V}_{0, \omega}(\Omega_\varepsilon) \times \mathbf{L}^2(\Omega_\varepsilon)/\mathbf{R}$.

Proposition 9 *The solution $(u^\varepsilon, p^\varepsilon)$ of (1) satisfies the following estimates*

$$\begin{aligned} \sup_{\varepsilon} \left(\int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right) &< +\infty, \\ \sup_{\varepsilon} \int_{\mathbf{R}^N} |u^\varepsilon|^2 dx &< +\infty, \\ \sup_{\varepsilon} \|p^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon)/\mathbf{R}} &< +\infty. \end{aligned}$$

Proof. 1. Taking u^ε as test-function in (8), we obtain

$$\begin{aligned} \nu \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \nu \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \\ = \int_{\Omega} f \cdot u^\varepsilon dx + \int_{\Sigma_\varepsilon} f \cdot u^\varepsilon dx \\ \leq \|f\|_{\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)} \|u^\varepsilon\|_{\mathbf{L}^1(\Omega_\varepsilon, \mathbf{R}^N)} \\ \leq \|f\|_{\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)} C(\Omega) \|\nabla u^\varepsilon\|_{\mathbf{L}^1(\Omega, \mathbf{R}^N)}, \end{aligned}$$

using Poincaré's inequality. Cauchy-Schwarz' inequality implies

$$\begin{aligned} \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \\ \leq C(f, \Omega) \left(\left(\int_{\Omega} |\nabla u^\varepsilon|^2 dx \right)^{1/2} + \left(\varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right)^{1/2} \right), \end{aligned}$$

whence, using the trivial inequality $(a + b)^2 \leq 2(a^2 + b^2)$

$$\int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \leq C \Rightarrow \|\nabla u^\varepsilon\|_{\mathbf{L}^1(\Omega_\varepsilon, \mathbf{R}^N)} \leq C.$$

The continuous embedding from $\mathbf{W}_{\Gamma_1}^{1,1}(\Omega_\varepsilon, \mathbf{R}^N)$ to $\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)$ implies the existence of a constant C independent of ε such that

$$\int_{\Omega_\varepsilon} |u^\varepsilon|^2 dx \leq C.$$

2. Let us define the zero mean value pressure $\overline{p^\varepsilon} = p^\varepsilon - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p^\varepsilon dx$, and let ψ_ε be the solution of the following problem (see [15])

$$\begin{cases} \operatorname{div}(\psi_\varepsilon) &= \overline{p^\varepsilon} & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon &= 0 & \text{on } \Gamma_1 \cup (\Gamma_2 \cap \omega), \\ \|\nabla \psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} &\leq C(\Omega) \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}, \end{cases} \quad (9)$$

for some constant $C(\Omega)$ independent of ε . Multiplying (1)_{1,2} by ψ_ε and using Green's formula, one obtains

$$\begin{aligned} & \nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx + \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon dx \\ &= \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon dx + \int_{\Omega_\varepsilon} (\overline{p^\varepsilon})^2 dx. \end{aligned}$$

Because

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon dx \right| &\leq \|f\|_{\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} \\ &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \\ \left| \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon dx \right| &\leq C \|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 \\ &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 \\ \left| \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx \right| &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}, \\ \left| \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx \right| &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}, \end{aligned}$$

thanks to (9)₃ and using Poincaré's inequality, we obtain

$$\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}^2 \leq C \left(\|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 + 1 \right) \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)},$$

which proves the third estimate. ■

Remark 10 We can observe that, when we impose an homogeneous Dirichlet boundary condition on the whole $\Gamma_{2,\varepsilon}$, for example when $\omega = \mathbf{R}^N$, the above estimates can be obtained in a simpler way, assuming only that $f \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$.

4 Convergence

Every function $u \in \mathbf{H}_{\Gamma_1}^1(\Omega_\varepsilon, \operatorname{div})$ can be extended in a function of the space $\mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$, still denoted u (see [16, Theorem 4.3.3], for example). We define the functional Φ^ε on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$ associated to (1), with the above-described modified boundary conditions on $\Gamma_{2,\varepsilon}$ through

$$\Phi^\varepsilon(u) = \begin{cases} \nu \int_{\Omega} |\nabla u|^2 dx + \nu \varepsilon \int_{\mathbf{R}^N \setminus \Omega} |\nabla u|^2 dx & \text{if } u \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div}), \\ +\infty & \text{otherwise} \end{cases} \quad (10)$$

and the functional Φ^0 defined on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$ through

$$\Phi^0(u) = \begin{cases} \nu \int_{\Omega} |\nabla u|^2 dx & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From the estimates given in Proposition 9, we can deduce that the asymptotic behaviour of the problem (1) is obtained when studying the Γ -limit of the associated energy functional for the following topology.

Definition 11 A sequence $(u_\varepsilon)_\varepsilon$ τ -converges to u , if it converges to u in the strong topology of $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$ and if $\sup_\varepsilon \Phi^\varepsilon(u_\varepsilon) < +\infty$.

We first present the Γ -convergence result for $(\Phi^\varepsilon)_\varepsilon$.

Proposition 12 When ε goes to 0, the sequence $(\Phi^\varepsilon)_\varepsilon$ Γ -converges to Φ^0 , in the topology τ .

Proof. Step 1: verification of the Γ -limsup. Take any $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and consider the set $\Omega^{0,\varepsilon} = \Omega \cup \partial\Omega \cup \Sigma^{0,\varepsilon}$, with

$$\Sigma^{0,\varepsilon} = \{x \in \mathbf{R}^N \mid 0 < d(x, \partial\Omega) < \sqrt{\varepsilon}\},$$

where $d(x, \partial\Omega)$ denotes the euclidean distance between x and the boundary $\partial\Omega$. Let $u^{1,\varepsilon}$ be such that $\operatorname{div}(u^{1,\varepsilon}) = 0$ in \mathbf{R}^N and

$$\|u - u^{1,\varepsilon}\|_{\mathbf{L}^2(\mathbf{R}^N \setminus \Omega^{0,\varepsilon}, \mathbf{R}^N)} < \varepsilon.$$

We define $\bar{u}^{1,\varepsilon}$ through

$$\bar{u}^{1,\varepsilon} = \begin{cases} u^{1,\varepsilon} & \text{in } \mathbf{R}^N \setminus \Omega^{0,\varepsilon}, \\ 0 & \text{on } \partial\Omega^{0,\varepsilon}. \end{cases}$$

We then take a nonnegative and smooth function $\rho_\varepsilon \in \mathbf{C}_c^\infty(\mathbf{R}^N)$ with support in $B(0, \varepsilon)$ and satisfying $\int_{\mathbf{R}^N} \rho_\varepsilon(x) dx = 1$. We define the function $\bar{u}^{0,\varepsilon}$ through $\bar{u}^{0,\varepsilon} = (\rho_\varepsilon * \bar{u}^{1,\varepsilon})|_{\mathbf{R}^N \setminus \overline{\Omega^{0,\varepsilon}}}$. There exists $\hat{u} \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$ such that $\operatorname{curl}(\hat{u}) = u$ in \mathbf{R}^N (see [15], for example). We finally define the function $u^{0,\varepsilon}$ through

$$u^{0,\varepsilon} = \begin{cases} \bar{u}^{0,\varepsilon} & \text{in } \mathbf{R}^N \setminus \overline{\Omega^{0,\varepsilon}}, \\ \operatorname{curl}\left(\hat{u} \frac{\sqrt{\varepsilon} - d(x, \partial\Omega)}{\sqrt{\varepsilon}}\right) & \text{in } \Sigma^{0,\varepsilon}, \\ u & \text{in } \Omega. \end{cases}$$

We immediately satisfy that $u^{0,\varepsilon} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$, that the sequence $(u^{0,\varepsilon})_\varepsilon$ converges to u in the topology τ and that

$$\limsup_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u^{0,\varepsilon}) \leq \nu \int_{\Omega} |\nabla u|^2 dx = \Phi^0(u).$$

Step 2: verification of the Γ -liminf. We take any sequence $(u_\varepsilon)_\varepsilon$ contained in $\mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$ which converges to u in the topology τ . We trivially have

$$\Phi^0(u) \leq \liminf_{\varepsilon \rightarrow 0} \Phi^0(u_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u_\varepsilon),$$

thanks to the lower semi-continuity property of Φ^0 for the weak topology of $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$. ■

We define the functional G^ε on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$ through

$$G^\varepsilon(u, \omega) = \begin{cases} \Phi^\varepsilon(u) + F^\varepsilon(u, \omega) & \text{if } u \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div}), \\ +\infty & \text{otherwise,} \end{cases}$$

where F^ε is defined in (4). Our main result is the following.

Theorem 13 There exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^N)$ and a symmetric matrix $\mu^\bullet = (\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having their support contained in Γ_2 , which are absolutely continuous with respect to the above-defined capacity Cap , and satisfying $\mu_{ij}(B) \zeta_i \zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$, $\forall B \in \mathcal{B}(\mathbf{R}^N)$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^N)$

$$\left(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G^\varepsilon\right)(u, \omega) = \nu \int_{\Omega} |\nabla u|^2 + \int_{\Gamma_2 \cap \omega} u_i u_j d\mu_{ij} =: G^0(u, \omega),$$

where the Γ -limit is taken with respect to the topology τ .

Proof. The upper and lower Γ -limits of the sequence $(G^\varepsilon)_\varepsilon$, with respect to the topology τ , exist, which are respectively defined through

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{B}(\mathbf{R}^N) : \begin{cases} G^s(u, B) &= \inf_{u_\varepsilon \xrightarrow{\tau} u} \limsup_{\varepsilon \rightarrow 0} G^\varepsilon(u_\varepsilon, B), \\ G^i(u, B) &= \inf_{u_\varepsilon \xrightarrow{\tau} u} \liminf_{\varepsilon \rightarrow 0} G^\varepsilon(u_\varepsilon, B). \end{cases} \quad (11)$$

Because F^ε takes nonnegative values and thanks to Proposition 12, we observe that, for every $B \in \mathcal{B}(\mathbf{R}^N)$, one has

$$G^s(., B) \geq \Phi^0(.) ; G^i(., B) \geq \Phi^0(.).$$

Let us define the functionals F^s and F^i on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{B}(\mathbf{R}^N)$ through

$$(F^0)^\alpha(u, B) = \begin{cases} G^\alpha(u, B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha = s, i$. Let $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and $(u_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$ be such that $(u_\varepsilon)_\varepsilon$ converges to u in the topology τ . We define $z_\varepsilon = u_\varepsilon - u$. Thus $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology τ . Replacing u_ε by $z_\varepsilon + u$ in (11), one obtains, using the quadratic property of Φ^ε

$$\begin{aligned} (F^0)^s(u, B) &= \inf_{z_\varepsilon \xrightarrow{\tau} 0} \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)), \\ (F^0)^i(u, B) &= \inf_{z_\varepsilon \xrightarrow{\tau} 0} \liminf_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)). \end{aligned}$$

The functionals $(F^0)^s$ and $(F^0)^i$ satisfy the following properties.

1. For every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$, $(F^0)^s(u, .)$ and $(F^0)^i(u, .)$ are nonnegative measures, because $F^\varepsilon(u + z_\varepsilon, .)$ is a measure for every $\varepsilon > 0$ and for every sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$ which converges to 0 in the topology τ .
2. $(F^0)^s(., B)$ and $(F^0)^i(., B)$ are lower semi-continuous on $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$, when equipped with its strong topology, because $G^s(., B)$, $G^i(., B)$ and Φ^0 are lower semi-continuous as upper, lower, or Γ -limits of functionals which are lower semi-continuous for this strong topology.
3. Let $\omega \in \mathcal{O}(\mathbf{R}^N)$ and $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$ be such that $u|_\omega = v|_\omega$. Then $(F^0)^s(u, \omega) = (F^0)^s(v, \omega)$ and $(F^0)^i(u, \omega) = (F^0)^i(v, \omega)$, because $F^\varepsilon(u + z_\varepsilon, \omega) = F^\varepsilon(v + z_\varepsilon, \omega)$, for every sequence $(z_\varepsilon)_\varepsilon$ such that $u + z_\varepsilon \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$, for every $\varepsilon > 0$.
4. Take any $\varphi \in \mathbf{C}^1(\mathbf{R}^N)$ such that $0 \leq \varphi \leq 1$, $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$ and $B \in \mathcal{B}(\mathbf{R}^N)$. One has, for every sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$ converging to 0 in the topology τ

$$\begin{aligned} F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi)v, B) &= F^\varepsilon((z_\varepsilon + u)\varphi + (1 - \varphi)(z_\varepsilon + v), B) \\ &\leq F^\varepsilon(z_\varepsilon + u, B) + F^\varepsilon(z_\varepsilon + v, B), \end{aligned}$$

because F^ε is \mathbf{C}^1 -convex. Because Φ^ε takes nonnegative values, for every $\varepsilon > 0$, one has

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi)v, B)) \\ &\leq \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B) + \Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)) \\ &\leq \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B)) \\ &\quad + \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)). \end{aligned}$$

Taking the infimum over all sequences $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ which converge to 0 in the topology τ , one obtains

$$(F^0)^s(\varphi u + (1 - \varphi)v, B) \leq (F^0)^s(u, B) + (F^0)^s(v, B).$$

We prove in a similar way that $(F^0)^s$ is convex. Thus $(F^0)^s$ is \mathbf{C}^1 -convex.

Thanks to the compacity theorem of [10], there exist a subsequence $(\varepsilon_k)_k$ and a dense and countable family $\mathcal{D} \subset \mathcal{B}(\mathbf{R}^N)$ such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{D}$

$$\left(\Gamma\text{-}\lim_{k \rightarrow +\infty} G^{\varepsilon_k} \right) (u, B) = G^0(u, B),$$

where the Γ -limit is taken with respect to the topology τ . We then define the functional F^0 on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}$ as

$$F^0(u, B) = \begin{cases} G^0(u, B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (12)$$

We have $F^0 = (F^0)^s = (F^0)^i$ on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}$. We then extend F^0 on $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{B}(\mathbf{R}^N)$ defining

$$F^0(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^i(u, D). \quad (13)$$

We define the family $\mathcal{R}(F)$ of Borel subsets of \mathbf{R}^N through

$$\mathcal{R}(F) = \left\{ B \in \mathcal{B}(\mathbf{R}^N) \mid \forall u \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) : (F^0)_+^s(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = \inf_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = (F^0)_-^s(u, B) \right\}.$$

Then we prove (see [5, Proposition 14.14]) that $\mathcal{R}(F^0)$ is a rich family in $\mathcal{B}(\mathbf{R}^N)$ and $F^0 = (F^0)^s = (F^0)_+^s = (F^0)_-^s = (F^0)_+^i = (F^0)_-^i = (F^0)^i$ on $\mathcal{R}(F^0)$. One obtains, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}(F^0)$

$$\begin{aligned} F^0(u, B) &= \inf_{z_{\varepsilon_k} \xrightarrow{\tau} 0} \limsup_{k \rightarrow +\infty} (\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u + z_{\varepsilon_k}, B)) \\ &= \inf_{z_{\varepsilon_k} \xrightarrow{\tau} 0} \liminf_{k \rightarrow +\infty} (\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u + z_{\varepsilon_k}, B)). \end{aligned}$$

Let now ε' denote any subsequence of ε . Thanks to the above method, there exist a subsequence $(\varepsilon'_k)_k$, a functional \mathcal{F}^0 and a rich family $\mathcal{R}(\mathcal{F}^0)$ such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}(\mathcal{F}^0)$

$$\begin{aligned} \mathcal{F}^0(u, B) &= \inf_{z_{\varepsilon'_k} \xrightarrow{\tau} 0} \limsup_{k \rightarrow +\infty} (\Phi^{\varepsilon'_k}(z_{\varepsilon'_k}) + F^{\varepsilon'_k}(u + z_{\varepsilon'_k}, B)) \\ &= \inf_{z_{\varepsilon'_k} \xrightarrow{\tau} 0} \liminf_{k \rightarrow +\infty} (\Phi^{\varepsilon'_k}(z_{\varepsilon'_k}) + F^{\varepsilon'_k}(u + z_{\varepsilon'_k}, B)). \end{aligned}$$

Because $\mathcal{R}(F^0) \cap \mathcal{R}(\mathcal{F}^0)$ is still a rich family, one has

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{R} : F^0(u, \cdot) = \mathcal{F}^0(u, \cdot), \text{ on } \mathcal{R}(F^0) \cap \mathcal{R}(\mathcal{F}^0).$$

Because the countable intersection of rich families is a rich family too, one can repeat the above reasoning and deduce the existence of a rich family \mathcal{R} in $\mathcal{B}(\mathbf{R}^N)$ on which the above limits coincide. One thus obtains, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}$

$$\left(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G^\varepsilon \right) (u, \omega) = \Phi^0(u) + F^0(u, B), \quad (14)$$

where the Γ -limit is taken with respect to the topology τ .

Thanks to the above properties 1., 2., 3. and 4. and to the relations (12) and (13), F^0 belongs to \mathbb{F} . Because Φ^ε and F^ε are quadratic, thanks to Corollary 7 and to Remark 8, there exist $\lambda \in \mathcal{M}_0$ finite, a

symmetric matrix $(a_{ij})_{i,j=1,\dots,N}$ of Borel functions from \mathbf{R}^N to \mathbf{R} with $a_{ij}(x)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$ and for q.e. $x \in \mathbf{R}^N$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^N)$

$$F^0(u, \omega) = \int_{\omega} u_i u_j d\mu_{ij},$$

with $\mu^\bullet = (\mu_{ij})_{i,j=1,\dots,N} = (a_{ij}\lambda)_{i,j=1,\dots,N} + \infty_{\mathbf{R}^N \setminus \Lambda} Id$, where Λ is defined as in Remark 8.

Let us now precise the support of μ^\bullet . For every $u, v \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$, such that $v|_{\Omega} = u|_{\Omega}$, one has

$$F^0(u, \mathbf{R}^N) = \int_{\mathbf{R}^N} v_i v_j d\mu_{ij},$$

because F^0 is local (\mathbf{R}^N belongs to \mathcal{R} because every rich family is dense, and every dense family contains \mathbf{R}^N). One deduces that $\text{supp}(\mu^\bullet) \subset \Omega \cup \Gamma_2$. Thanks to (14), one has

$$0 \leq \int_{\mathbf{R}^N} u_i u_j d\mu_{ij} + \Phi^0(u) \leq \liminf_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(u) + F^\varepsilon(u, \mathbf{R}^N)). \quad (15)$$

Taking $u \in \mathbf{H}_0^1(\Omega, \text{div}) = \{u \in \mathbf{H}_0^1(\Omega, \mathbf{R}^N) \mid \text{div}(u) = 0\}$, then, for every $\varepsilon > 0$, $F^\varepsilon(u, \mathbf{R}^N) = 0$, and $\liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u) = \Phi^0(u)$. One deduces, using (15), that $\int_{\Omega} u_i u_j d\mu_{ij} = 0$, and thus that $\text{supp}(\mu^\bullet) \subset \Gamma_2$, which ends the proof. ■

Remark 14 1. We thus get Navier's wall law at the zeroth-order limit of the problem (1).

2. Theorem 13 can be extended to every kind of obstacle functional in \mathbb{F} , using Theorem 6 for the integral representation. One can define, for example, sequences of obstacle functionals on $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$ of the kind

$$(F^\varepsilon)^+(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} \geq 0 \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise,} \end{cases}$$

the limit $(F^0)^+$ of which is defined on $\mathbf{V}_{\Gamma_1}(\Omega) \times (\mathcal{R}^+ \cap \mathcal{O}(\mathbf{R}^N))$ (for some rich family \mathcal{R}^+) as

$$(F^0)^+(u, \omega) = \int_{\omega \cap \Gamma_2} u_i^+ u_j^+ d\mu_{ij},$$

where $u_i^+ = \max(0, u_i)$, $i = 1, \dots, N$.

3. One proves that $\mu_{ij} \in \mathbf{H}^{-1/2}(\Gamma_2)$, $\forall i, j = 1, \dots, N$, where μ_{ij} is the measure defined in Theorem 13. One first observes that the measure λ defined in Theorem 6 belongs to $\mathbf{H}^{-1/2}(\Gamma_2)^+$. λ is indeed finite. Because for every compact subset $K \subset \Gamma_2$, one has $\lambda(K) < +\infty$, hence λ is a Radon nonnegative measure. Moreover, because λ is absolutely continuous with respect to the capacity Cap , we deduce from [6, Theorem 2.2], the existence of a Radon measure $\varkappa \in \mathbf{H}^{-1/2}(\Gamma_2)$ and of a Borel function $f : \Gamma_2 \rightarrow [0, +\infty[$ such that $f = \frac{d\lambda}{d\varkappa}$.

Let us come back to the study of problem (1). The solution u^ε of (1), with the homogeneous Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ is also the solution of the minimization problem

$$\inf_{v \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \left(G^\varepsilon(v, \mathbf{R}^N) + 2 \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v dx - 2 \int_{\Omega_\varepsilon} f \cdot v dx \right). \quad (16)$$

From Theorem 13, one deduces the following asymptotic behaviour of the solution of (1).

Corollary 15 *The solution $(u^\varepsilon, p^\varepsilon)$ of (1), is such that $(u^\varepsilon)_\varepsilon$ converges to u^0 in the topology τ and $((p^\varepsilon)|_\Omega)_\varepsilon$ converges to p^0 in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$, where (u^0, p^0) belongs to $\mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$ and is the solution of the limit minimization problem*

$$v \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \left(G^0(v, \mathbf{R}^N) + 2 \int_\Omega (u^0 \cdot \nabla) u^0 \cdot v dx - 2 \int_\Omega f \cdot v dx \right), \quad (17)$$

or of the limit problem with Navier law

$$\begin{cases} -\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 &= f & \text{in } \Omega, \\ \operatorname{div}(u^0) &= 0 & \text{in } \Omega, \\ u^0 &= 0 & \text{on } \Gamma_1, \\ u^0 \cdot n &= 0 & \text{on } \Gamma_2, \\ (I - n \otimes n) \nu \frac{\partial u^0}{\partial n} + \mu^\bullet u^0 &= 0 & \text{on } \Gamma_2. \end{cases} \quad (18)$$

Proof. We first observe that, for every sequence $(v_\varepsilon)_\varepsilon$ converging to v in the topology τ

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f \cdot v_\varepsilon dx = \int_\Omega f \cdot v dx,$$

Thanks to the properties of the Γ -convergence, $(u^\varepsilon)_\varepsilon$ converges to u^0 in the topology τ , with $u^0 \in \mathbf{V}_{\Gamma_1}(\Omega)$, and

$$\lim_{\varepsilon \rightarrow 0} G^\varepsilon(u^\varepsilon, \mathbf{R}^N) = G^0(u^0, \mathbf{R}^N) = \nu \int_\Omega |\nabla u^0|^2 dx + \int_{\Gamma_2} (u^0)_i (u^0)_j d\mu_{ij}.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v_\varepsilon dx = \int_\Omega (u^0 \cdot \nabla) u^0 \cdot v dx,$$

for every sequence $(v_\varepsilon)_\varepsilon$ converging to v in the topology τ . For every $\varphi \in \mathbf{C}^1(\mathbf{R}^N)$, one has

$$\left| \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx \right| \leq \left(\int_{\Sigma_\varepsilon} |\nabla \varphi|^2 dx \right)^{1/2} \left(\int_{\mathbf{R}^N} |u^\varepsilon|^2 dx \right)^{1/2},$$

and thus $\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = 0$. Because $\operatorname{div}(u^\varepsilon) = \operatorname{div}(u^0) = 0$, and $u^\varepsilon = 0$, q.e. on Γ_2 , one has

$$0 = \int_{\Omega_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = \int_\Omega u^\varepsilon \cdot \nabla \varphi dx + \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx.$$

Taking the limit of this equality, we obtain

$$0 = \int_\Omega u^0 \cdot \nabla \varphi dx = \int_{\Gamma_2} u^0 \cdot n \varphi d\Gamma_2,$$

which proves that $u^0 \cdot n = 0$ on Γ_2 . Thus $u^0 \in \mathbf{V}_{0,\Gamma_1}(\Omega)$ is the solution of the problem (17). The variational formulation of (17) can be written as

$$\begin{aligned} \forall \varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega) : \int_\Omega (-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0) \cdot \varphi dx \\ + \int_{\Gamma_2} \nu \frac{\partial u^0}{\partial n} \cdot \varphi d\Gamma_2 + \int_{\Gamma_2} (u^0)_i \varphi_j d\mu_{ij} = \int_\Omega f \cdot \varphi dx. \end{aligned}$$

There exists $p_0 \in L^2(\Omega)/\mathbf{R}$ such that $-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 - f = -\nabla p_0$. Thanks to Proposition 9, the sequence $((p^\varepsilon)|_\Omega)_\varepsilon$ converges to p^0 in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$. Because $\varphi \cdot n = 0$ on Γ_2 , with $n = (0, 0, 1)$, one has: $\nu \frac{\partial u^0}{\partial n} \cdot \varphi = (Id - n \otimes n) \nu \frac{\partial u^0}{\partial n} \cdot \varphi$, which ends the proof. ■

5 Special cases

We intend to specialize the general result obtained in Theorem 13, in two cases where the boundary $\Gamma_{2,\varepsilon}$ can be defined through some Lipschitz continuous function.

5.1 Periodic case

In this section, we suppose that $\Omega \subset \{x_3 > 0\}$ with $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$, Γ_2 containing 0. We define $Y = (-1/2, 1/2)^2$ and consider a Y -periodic function $h \in \mathbf{C}_c^2(Y, \mathbf{R}_+)$. For every $k \in \mathbf{Z}^2$, we define $Y_\varepsilon^k = (-\varepsilon/2, \varepsilon/2)^2 + (k_1\varepsilon, k_2\varepsilon)$, and let $I_\varepsilon = \{k \in \mathbf{Z}^2 \mid Y_\varepsilon^k \subset \Gamma_2\}$. We define h_ε on Γ_2 through

$$h_\varepsilon(x') = \begin{cases} h\left(\frac{x'}{\varepsilon}\right) & \text{if there exists } k \in I_\varepsilon \text{ such that } x' = (x_1, x_2) \in Y_\varepsilon^k, \\ 0 & \text{otherwise} \end{cases}$$

and Σ_ε through

$$\Sigma_\varepsilon = \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon h_\varepsilon(x') < x_3 < 0\}.$$

Thanks to Theorem 13, there exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^3)$, a symmetric matrix $(\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having the same support contained in Γ_2 , absolutely continuous with respect to the capacity Cap , and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^3$, $\forall B \in \mathcal{B}(\mathbf{R}^3)$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^3)$

$$\inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\} \cap \omega \text{ and } z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} = \int_{\omega \cap \Gamma_2} u_i u_j d\mu_{ij}, \quad (19)$$

where Φ^ε is the energy functional defined in (10).

Because the lower boundary $\Gamma_{2,\varepsilon}$ of Σ_ε , defined through the equality $\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h_\varepsilon(x')\}$, has a periodic structure, the measures μ_{ij} , $i, j = 1, \dots, N$, are invariant under translations on Γ_2 . This implies $\mu_{ij} = K_{ij} dx'$, where K_{ij} , $i, j = 1, 2, 3$, are constants in $\overline{\mathbf{R}}$ satisfying $K_{ij}\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^3$.

The purpose of this section is to identify these constants K_{ij} , $i, j = 1, 2, 3$. We observe that we do not have to determine K_{i3} , $i = 1, 2, 3$, because, in the limit problem, one has $u \cdot n = u \cdot e_3 = u_3 = 0$.

Theorem 16 *The limit Navier wall law of the limit problem (18) is in this case*

$$\frac{\partial(u^0)_m}{\partial x_3} = c_m (u^0)_m, \text{ on } \Gamma_2, m = 1, 2,$$

where the constants c_m are defined in (21).

Proof. We define the set $Z_h = \{x \mid x' \in Y, -h(x') < x_3 < 0\}$ and consider in Z_h the local Stokes problems for $m = 1, 2$

$$\begin{cases} -\Delta w^m + \nabla q^m &= e^m & \text{in } Z_h, \\ \operatorname{div}(w^m) &= 0 & \text{in } Z_h, \\ w^m &= e^m & \text{on } \{x_3 = -h(x')\}, \\ w^m &= 0 & \text{on } \{x_3 = 0\}, \\ w^m, q^m && Y\text{-periodic,} \end{cases} \quad (20)$$

where e^m is the m -th vector of the canonical basis of \mathbf{R}^3 . Lax-Milgram' Theorem implies that (20) has a unique solution (w^m, q^m) with

$$\begin{aligned} w^m &\in \mathbf{V}(Z_h) = \left\{ u \in \mathbf{H}^1(Z_h, \mathbf{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } Z_h, \right. \\ &\quad \left. u = 0 \text{ on } \{x_3 = 0\}, u \text{ } Y\text{-periodic} \right\} \\ q^m &\in \mathbf{L}^2(Z_h)/\mathbf{R}, q^m \text{ } Y\text{-periodic.} \end{aligned}$$

Let $z_h = \max_{x' \in Y} h(x')$ and choose $H > z_h$. We define

$$\tilde{Z}_h = \{x \mid x' \in Y, -H < x_3 < -h(x')\}$$

and consider in \tilde{Z}_h problems similar to (20) except that we impose $\tilde{w}^m = e^m$ on $\{x_3 = -h(x')\}$ and $\tilde{w}^m = 0$ on $\{x_3 = -H\}$. Let us define

$$\begin{aligned} \tilde{\Sigma}_\varepsilon &= \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < -\varepsilon h_\varepsilon(x')\}, \\ B_\varepsilon &= \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < 0\} \end{aligned}$$

and the functions $(w^{\varepsilon m}, q^{\varepsilon m})$ and $(\tilde{w}^{\varepsilon m}, \tilde{q}^{\varepsilon m})$ through

$$\begin{cases} w^{\varepsilon m}(x) &= w^m\left(\frac{x}{\varepsilon}\right), & q^{\varepsilon m}(x) &= q^m\left(\frac{x}{\varepsilon}\right), \\ \tilde{w}^{\varepsilon m}(x) &= \tilde{w}^m\left(\frac{x}{\varepsilon}\right), & \tilde{q}^{\varepsilon m}(x) &= \tilde{q}^m\left(\frac{x}{\varepsilon}\right). \end{cases}$$

We finally build the function z_ε^{0m} , on B_ε , through

$$z_\varepsilon^{0m}(x) = \begin{cases} w^{\varepsilon m}(x) & \text{if } x \in \Sigma_\varepsilon, \\ e^m & \text{on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \\ \tilde{w}^{\varepsilon m}(x) & \text{on } \tilde{\Sigma}_\varepsilon. \end{cases}$$

Because $h = 0$ on ∂Y , one can suppose that $z_\varepsilon^{0m} = 0$ on $\partial\Gamma_2 \times (-\varepsilon H, 0)$. This implies that $z_\varepsilon^{0m} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \text{div})$ and $z_\varepsilon^{0m} = 0$ on ∂B_ε . Moreover

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} |z_\varepsilon^{0m}|^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |z_\varepsilon^{0m}|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{-\varepsilon H}^0 |z_\varepsilon^{0m}|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} \left(\sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-H}^{-h(x')} |\tilde{w}^m(x)|^2 dx \right. \\ &\quad \left. + \sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-h(x')}^0 |w^m(x)|^2 dx \right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) &= \lim_{\varepsilon \rightarrow 0} \nu \varepsilon \int_{\Sigma_\varepsilon} |\nabla z_\varepsilon^{0m}|^2 dx \\ &= \nu \sum_{k \in I_\varepsilon} \varepsilon^2 \int_Y \int_{-h(x')}^0 |\nabla w^m(x)|^2 dx \\ &= \nu |\Gamma_2| c_m, \end{aligned}$$

with

$$c_m = \int_{Z_h} |\nabla w^m|^2 dx. \quad (21)$$

Taking $u = -e^m$ on Σ_ε , in (19), one obtains

$$\begin{aligned} K_{mm} |\Gamma_2| &= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^m \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \right. \\ &\quad \left. z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) = \nu c_m |\Gamma_2|. \end{aligned}$$

This implies

$$K_{mm} |\Gamma_2| \leq \nu c_m |\Gamma_2|. \quad (22)$$

Take any sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \text{div})$ such that $z_\varepsilon = e^m$ on the surface $\{x_3 = -\varepsilon h_\varepsilon(x')\}$ and $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology τ . We write the subdifferential inequality

$$\Phi^\varepsilon(z_\varepsilon) \geq \Phi^\varepsilon(z_\varepsilon^{0m}) + 2\nu\varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx. \quad (23)$$

We observe that

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx &= -\varepsilon \int_{\Sigma_\varepsilon} \Delta z_\varepsilon^{0m} \cdot (z_\varepsilon - z_\varepsilon^{0m}) dx \\ &\quad - \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^{0m}) d\Gamma_2. \end{aligned}$$

Using the regularity (at least \mathbf{H}^2) of w^m , we obtain

$$\varepsilon \Delta z_\varepsilon^{0m} \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_{\Gamma_2} \int_{Z_h} \Delta w^m(x) dx,$$

where the convergence takes place in the weak topology of $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$ and $\mathbf{1}_{\Gamma_2}$ is the characteristic function of Γ_2 . Then

$$\begin{aligned} &\left| \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^{0m}) d\Gamma_2 \right| \\ &\leq \left(\int_{\Gamma_2} \left| \frac{\partial w^m}{\partial n} \right|^2 d\Gamma_2 \right)^{1/2} \left(\int_{\mathbf{R}^3} |z_\varepsilon - z_\varepsilon^{0m}|^2 dx \right)^{1/2}. \end{aligned}$$

Because $(z_\varepsilon - z_0^{\varepsilon m})_\varepsilon$ converges to 0 in the strong topology $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx = 0.$$

Taking the \liminf in (23), one obtains

$$\liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) = \nu c_m |\Gamma_2|.$$

In this last inequality, taking the infimum with respect to all sequences $(z_\varepsilon)_\varepsilon$ satisfying the imposed conditions, one obtains: $K_{mm} |\Gamma_2| \geq \nu c_m |\Gamma_2|$. This inequality and (22) imply: $K_{mm} = \nu c_m$. Taking now $u = -(e^1 + e^2)$ on Σ_ε in (19), one obtains

$$\begin{aligned} (K_{11} + 2K_{12} + K_{22}) |\Gamma_2| &= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^1 + e^2 \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{01} + z_\varepsilon^{02}). \end{aligned}$$

Because $\int_{Z_h} \nabla w^1 \cdot \nabla w^2 dz = 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{01} + z_\varepsilon^{02}) = \nu |\Gamma_2| (c_1 + c_2).$$

This implies: $K_{12} \leq 0$, through the above expression of K_{mm} . Writing a subdifferential inequality as in (23), one obtains: $K_{12} \geq 0$, which implies: $K_{12} = 0$. ■

5.2 Case where h_ε is independent of ε

As in the previous section, we still suppose that $\Omega \subset \{x_3 > 0\}$ and $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$. But, we here suppose that the boundary $\Gamma_{2,\varepsilon}$ is given as

$$\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h(x')\}$$

where h is a Lipschitz continuous function satisfying $h(x') > 0, \forall x' \in \Gamma_2$. We have the following result.

Theorem 17 *Under the preceding hypothesis, the Navier wall law is in this case*

$$(Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{h} = 0, \text{ on } \Gamma_2.$$

Proof. Thanks to Theorem 13, there exist a rich family $\mathcal{R}_{\Gamma_2} \subset \mathcal{B}(\Sigma)$, a symmetric matrix $(\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having their support contained in Γ_2 , which are absolutely continuous with respect to the capacity Cap , and satisfying $\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^3, \forall B \in \mathcal{B}(\Sigma)$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$

$$\int_{\omega} u_i u_j d\mu_{ij} = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on } \{x_3 = -\varepsilon h(x')\} \cap \omega, z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\}. \quad (24)$$

Take $u = -e^1$ on $\{x_3 = -\varepsilon h(x')\}$. Then choose $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$, an open subset ω^ε of \mathbf{R}^2 such that $\omega^\varepsilon \setminus \bar{\omega} = \{x' \in \mathbf{R}^2 \mid 0 < d(x', \partial\omega) < \varepsilon\}$ and $\varphi^\varepsilon \in \mathbf{C}^1(\mathbf{R}^2)$ with $0 \leq \varphi^\varepsilon \leq 1$ such that

$$\begin{cases} \varphi^\varepsilon &= 1 & \text{in } \omega, \\ \varphi^\varepsilon &= 0 & \text{on } \partial\omega_\varepsilon. \end{cases}$$

We define the function $w^{1\varepsilon}$ through

$$\begin{cases} (w^{1\varepsilon})_1(x) &= \frac{x_3}{\varepsilon h(x')} \varphi^\varepsilon(x'), \\ (w^{1\varepsilon})_2(x) &= 0, \\ (w^{1\varepsilon})_3(x) &= \frac{\varepsilon}{2} \left(\frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') h(x') \right) \\ &\quad + \frac{(x_3)^2}{2} \left(\frac{1}{\varepsilon h(x')} \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') - \frac{\varphi^\varepsilon(x')}{\varepsilon h^2(x')} \frac{\partial h}{\partial x_1}(x') \right). \end{cases}$$

One has $\operatorname{div}(w^{1\varepsilon}) = 0, \forall \varepsilon > 0$, and $w^{1\varepsilon} = e^1$ on $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$. We now consider the problem

$$\begin{cases} -\Delta \zeta^{1\varepsilon} + \nabla \varpi^{1\varepsilon} &= e^1 & \text{in } \Omega, \\ \operatorname{div}(\zeta^{1\varepsilon}) &= 0 & \text{in } \Omega, \\ \zeta^{1\varepsilon} &= 0 & \text{on } \Gamma_1, \\ \zeta^{1\varepsilon} &= \left(0, 0, \frac{1}{2} \left(\frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') h(x') \right) \right) & \text{on } \Gamma_2. \end{cases} \quad (25)$$

The problem (25) has a unique solution $(\zeta^{1\varepsilon}, \varpi^{1\varepsilon}) \in \mathbf{H}_{\Gamma_1}^1(\Omega, \operatorname{div}) \times \mathbf{L}^2(\Omega)/\mathbf{R}$, satisfying

$$\int_{\Omega} |\nabla \zeta^{1\varepsilon}|^2 dx \leq C; \int_{\Omega} |\zeta^{1\varepsilon}|^2 dx \leq C,$$

where C is a constant independent of ε . Let $H > z_h$, with $z_h = \max_{\Gamma_2} h$. We define the function $\tilde{w}^{1\varepsilon}$ in $D_\varepsilon = \{x \mid -H < x_3 < -\varepsilon h(x')\}$ through

$$\left\{ \begin{array}{lcl} (\tilde{w}^{1\varepsilon})_1(x) & = & \frac{x_3 + H}{\varepsilon(H - h(x'))} \varphi^\varepsilon(x'), \\ (\tilde{w}^{1\varepsilon})_2(x) & = & 0, \\ (\tilde{w}^{1\varepsilon})_3(x) & = & \frac{\varepsilon}{2} \left(\frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') (H - h(x')) \right) \\ & & - \frac{(x_3 + H)^2}{2} \left(\frac{1}{\varepsilon(H - h(x'))} \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') \right. \\ & & \left. + \frac{\varphi^\varepsilon(x')}{\varepsilon(H - h(x'))^2} \frac{\partial h}{\partial x_1}(x') \right). \end{array} \right.$$

We consider the bounded, smooth and open subset $\Omega_H = \{x \mid x_3 > -H\}$ and $\partial\Omega_H \cap \{x \mid x_3 = -H\} = \Gamma_2$, and the solution $(\zeta_H^{1\varepsilon}, \omega_H^{1\varepsilon})$ of the problem

$$\left\{ \begin{array}{ll} -\Delta \zeta_H^{1\varepsilon} + \nabla \varpi_H^{1\varepsilon} & = e^1 \quad \text{in } \Omega_H, \\ \operatorname{div}(\zeta_H^{1\varepsilon}) & = 0 \quad \text{in } \Omega_H, \\ \zeta_H^{1\varepsilon} & = 0 \quad \text{on } \Omega_H \setminus \Gamma_2, \\ \zeta_H^{1\varepsilon} & = \left(0, 0, \frac{1}{2} \left(\frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') (H - h(x')) \right) \right) \quad \text{on } \Gamma_2. \end{array} \right.$$

Let us define the function $z_0^{1,\varepsilon}$ through

$$z_\varepsilon^{0,1} = \begin{cases} \varepsilon \zeta^{1\varepsilon} & \text{in } \Omega, \\ w^{1\varepsilon} & \text{in } \Sigma_\varepsilon, \\ \tilde{w}^{1\varepsilon} & \text{in } D_\varepsilon, \\ \varepsilon \zeta_H^{1\varepsilon} & \text{in } \Omega_H. \end{cases}$$

One immediately verifies that $z_\varepsilon^{0,1} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \operatorname{div})$, $z_\varepsilon^{0,1} = e^1$ on the surface $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$, $(z_\varepsilon^{0,1})_\varepsilon$ converges to 0 in the strong topology of $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$ and

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0,1}) = \lim_{\varepsilon \rightarrow 0} \nu \varepsilon \int_{\omega \times (-\varepsilon h(x'), 0)} |\nabla z_\varepsilon^{0,1}|^2 dx = \nu \int_\omega \frac{dx'}{h(x')}.$$

One thus deduces from (24) within this context

$$\mu_{11}(\omega) \leq \nu \int_\omega \frac{dx'}{h(x')}.$$

Furthermore, taking $(z_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \operatorname{div})$, $z_\varepsilon = e^1$ on $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$, $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology τ , and using the subdifferential inequality

$$\begin{aligned} \Phi^\varepsilon(z_\varepsilon) &\geq \Phi^\varepsilon(z_\varepsilon^{0,1}) \\ &+ \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0,1} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0,1}) dx + \nu \int_\Omega \nabla z_\varepsilon^{0,1} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0,1}) dx, \end{aligned}$$

we prove that $\mu_{11}(\omega) \geq \nu \int_\omega dx'/h(x')$. This implies the equality: $\mu_{11}(\omega) = \nu \int_\omega dx'/h(x')$ and, since this equality is true for every $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$, we obtain $\mu_{11} = \nu dx'/h(x')$.

Choosing now $u = -e^2$ on Σ_ε , we can build a test-function $z_\varepsilon^{0,2}$ in a similar way and prove: $\mu_{22} = \nu dx'/h(x')$.

Finally, taking $u = -(e^1 + e^2)$ on Σ_ε , we consider the sequence $(z_\varepsilon^0)_\varepsilon$ defined through: $z_\varepsilon^0 = z_\varepsilon^{0,1} + z_\varepsilon^{0,2}$. One deduces from the above computations that

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^0) = \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0,1} + z_\varepsilon^{0,2}) = 2\nu \int_\omega \frac{dx'}{h(x')}$$

and, as in the periodic case, that $\mu_{12} = 0$. The boundary conditions on Γ_2 can thus be written as

$$\begin{cases} (u^0)_3 = 0, \\ \frac{\partial (u^0)_m}{\partial x_3} = \frac{1}{h} (u^0)_m, \quad m = 1, 2, \end{cases}$$

which ends the proof. ■

Remark 18 In a general way, if $\Sigma_\varepsilon = \{\sigma + tn \mid \sigma \in \Gamma_2, -\varepsilon h(\sigma) < t < 0\}$, with h positive and Lipschitz continuous on Γ_2 , we can prove that the limit law is

$$\begin{cases} (Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{h} = 0, \\ u^0 \cdot n = 0. \end{cases}$$

6 Optimal control problem

For a given real $m > 0$, we consider the set Ξ_m of all matrices $\mathbf{h} = \text{Diag}(h_i)_{i=1,\dots,N}$ of functions $h_i : \Gamma_2 \rightarrow [0, +\infty]$, $d\Gamma_2$ -measurable and such that

$$\int_{\Gamma_2} h_i d\Gamma_2 = m, \quad \forall i = 1, \dots, N.$$

We suppose that $\partial\Omega$ is \mathbf{C}^2 and consider the Navier-Stokes problem, with Navier wall law, according to Theorem 17

$$\begin{cases} -\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h = f & \text{in } \Omega, \\ \operatorname{div}(u^h) = 0 & \text{in } \Omega, \\ \mathbf{h}(Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h = 0 & \text{on } \Gamma_2, \\ u^h \cdot n = 0 & \text{on } \Gamma_2, \\ u^h = 0 & \text{on } \Gamma_1, \end{cases} \quad (26)$$

which has a unique solution $(u^h, p^h) \in \mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$. We define the functional \mathbf{F} defined on $\Xi_m \times \mathbf{H}_{\Gamma_1}^1(\Omega, \operatorname{div})$ and associated to (26) through

$$\mathbf{F}(\mathbf{h}, u) = \begin{cases} \frac{\nu}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 \\ \quad + \int_\Omega (u^h \cdot \nabla) u^h \cdot u dx - \int_\Omega f \cdot u dx & \text{if } u \in \mathbf{V}_{0,\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the optimal control problem (3), which means that the cost functional is here taken as the global energy. We observe that

$$\mathbf{F}(\mathbf{h}, u^h) = - \int_\Omega f \cdot u^h dx.$$

This implies that the minimization of \mathbf{F} , with respect to u on the set $\mathbf{V}_{0,\Gamma_1}(\Omega)$, is equivalent to the maximization of the work of the external forces on this set. The problem (3) has a unique minimizer when Poincaré's inequality

$$\left(\int_{\Gamma_2} |u_i| d\Gamma_2 \right)^2 \leq \int_{\Gamma_2} h_i d\Gamma_2 \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2,$$

becomes an equality, for every $i = 1, \dots, N$, that is when

$$h_i^m = m \frac{|u_i^m|_{\Gamma_2}}{\int_{\Gamma_2} |u_i^m| d\Gamma_2},$$

where (u^m, p^m) is the solution of

$$\left\{ \begin{array}{l} -\nu \Delta u^m + (u^m \cdot \nabla) u^m + \nabla p^m = f \quad \text{in } \Omega, \\ \operatorname{div}(u^m) = 0 \quad \text{in } \Omega, \\ u^m \cdot n = 0 \quad \text{on } \Gamma_2, \\ u^m = 0 \quad \text{on } \Gamma_1, \\ \\ (Id - n \otimes n) \frac{\partial u^m}{\partial n} \\ + \frac{1}{m} \left(\begin{array}{c} \operatorname{sign}((u^m)_1(x)) \int_{\Gamma_2} |(u^m)_1| d\Gamma_2 \\ \vdots \\ \operatorname{sign}((u^m)_N(x)) \int_{\Gamma_2} |(u^m)_N| d\Gamma_2 \end{array} \right) = 0 \quad \text{on } \Gamma_2. \end{array} \right.$$

Trivially, the study of the Γ -convergence of the sequence of the energies associated to (3), when m goes to 0 and relatively to the weak topology of $\mathbf{H}^1(\Omega, \mathbf{R}^N)$, will lead to the following conclusions: $(u^m)_m$ converges to u^0 in the weak topology of $\mathbf{H}^1(\Omega, \mathbf{R}^N)$, $(p^m)_m$ converges to p^0 in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$, where (u^0, p^0) is the solution of the problem

$$\left\{ \begin{array}{l} -\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f \quad \text{in } \Omega, \\ \operatorname{div}(u^0) = 0 \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (27)$$

In order to study the asymptotic behavior of $\left((u^m/m)_{|\Gamma_2}\right)_m$, we introduce the following linearized perturbation of the Navier-Stokes problem (27)

$$\left\{ \begin{array}{l} -\nu \Delta u^{0,m} + \nabla p^{0,m} = f - (u^m \cdot \nabla) u^m \quad \text{in } \Omega, \\ \operatorname{div}(u^{0,m}) = 0 \quad \text{in } \Omega, \\ u^{0,m} = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (28)$$

The problem (28) is a Stokes system, the source term of which is $f - (u^m \cdot \nabla) u^m$. Consider now the functional I_m defined on $\mathbf{V}_{0,\Gamma_1}(\Omega)$ through

$$\begin{aligned} I_m(v) &= \frac{m\nu}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{1}{2} \sum_{i=1}^N \left(\int_{\Gamma_2} |v_i| d\Gamma_2 \right)^2 \\ &\quad + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot v d\Gamma_2. \end{aligned}$$

I_m has a unique minimizer $(v^m, q^m) \in \mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$ which is the solution of the problem

$$\left\{ \begin{array}{lcl} -\nu m \Delta v^m + \nabla q^m & = & 0 \quad \text{in } \Omega, \\ \operatorname{div}(v^m) & = & 0 \quad \text{in } \Omega, \\ v^m \cdot n & = & 0 \quad \text{on } \Gamma_2, \\ v^m & = & 0 \quad \text{on } \Gamma_1, \\ (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} + m (Id - n \otimes n) \frac{\partial v^m}{\partial n} \\ + \begin{pmatrix} \operatorname{sign}((v^m)_1) \int_{\Gamma_2} |(v^m)_1| d\Gamma_2 \\ \vdots \\ \operatorname{sign}((v^m)_N) \int_{\Gamma_2} |(v^m)_N| d\Gamma_2 \end{pmatrix} & = & 0 \quad \text{on } \Gamma_2. \end{array} \right.$$

We observe that the couple (v^m, q^m) defined through

$$v^m = \frac{u^m - u^{0,m}}{m}; \quad q^m = p^m - p^{0,m},$$

is the minimizer of I_m . For every $\varphi \in \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N)$, there exists a unique extension $v_\varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega)$ of φ defined through

$$\int_{\Omega} |\nabla v_\varphi|^2 dx = \inf_{\{w \in \mathbf{V}_{0,\Gamma_1}(\Omega) | w|_{\Gamma_2} = \varphi\}} \int_{\Omega} |\nabla w|^2 dx.$$

Let us denote $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ the space of finite Radon measures on Γ_2 with values in \mathbf{R}^N . We consider the functional J_m defined on $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ through

$$J_m(\varphi) = \begin{cases} \frac{m\nu}{2} \int_{\Omega} |\nabla v_\varphi|^2 dx + \frac{1}{2} \sum_{i=1}^N \left(\int_{\Gamma_2} |\varphi_i| d\Gamma_2 \right)^2 \\ + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot \varphi d\Gamma_2 & \text{if } \varphi \in \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N) \\ & \text{and } \varphi \cdot n = 0 \text{ on } \Gamma_2, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $(v^m)|_{\Gamma_2}$ is the unique minimizer of J_m .

Proposition 19 *One has the following properties.*

1. $\sup_m \sum_{i=1}^n \left(\int_{\Gamma_2} |v_i^m| d\Gamma_2 \right) < +\infty$.
2. The sequence $(J_m)_m$ Γ -converges, when m tends to 0 and with respect to the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$, to the functional J defined from $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ to \mathbf{R} through

$$J(\lambda) = \sum_{i=1}^N (|\lambda_i|(\Gamma_2))^2 + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^0}{\partial n} d\lambda,$$

where $|\lambda_i|(\Gamma_2)$ is the total variation of λ_i on Γ_2 .

Proof. 1. Remark that a regularity property of the boundary $\partial\Omega$ implies that

$$\sup_m \left\| (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \right\|_{\mathbf{L}^\infty(\Gamma_2, \mathbf{R}^N)} < +\infty.$$

One thus obtains

$$J_m \left((v^m)_{|\Gamma_2} \right) \geq \frac{1}{2} \sum_{i=1}^N \left(\int_{\Gamma_2} |v_i^m| d\Gamma_2 \right)^2 - \frac{C}{2} \sum_{i=1}^N \left(\int_{\Gamma_2} |v_i^m| d\Gamma_2 \right).$$

Moreover

$$\sup_m J_m \left((v^m)_{|\Gamma_2} \right) \leq \sup_m J_m(0) = 0 \Rightarrow \sup_m \sum_{i=1}^N \left(\int_{\Gamma_2} |v_i^m| d\Gamma_2 \right) \leq C.$$

This implies the existence of a subsequence of $\left((v^m)_{|\Gamma_2} \right)_m$, still denoted $\left((v^m)_{|\Gamma_2} \right)_m$, which converges to some λ in the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$.

2. Choose any sequence $(\varphi^m)_m \subset \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N)$, satisfying $\varphi^m \cdot n = 0$, on Γ_2 and converging to λ in the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$. The functional $\mu \mapsto |\mu|$, where $|\mu|$ is the total variation of μ , being lower semi-continuous on $\mathcal{M}(\Gamma_2)$, one has

$$\liminf_{m \rightarrow 0} \int_{\Gamma_2} |\varphi_i^m| d\Gamma_2 \geq |\lambda_i|(\Gamma_2).$$

Thanks to the regularity of the boundary, $\left((Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \right)_m$ uniformly converges to $(Id - n \otimes n) \frac{\partial u^0}{\partial n}$, hence

$$\liminf_{m \rightarrow 0} \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot \varphi^m d\Gamma_2 \geq \int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\lambda_i.$$

This implies

$$\liminf_{m \rightarrow 0} J_m(\varphi^m) \geq J(\lambda). \quad (29)$$

In order to prove the Γ -lim sup property, let us suppose that $\Omega \subset \{x_N < 0\}$ and $\partial\Omega \cap \{x_N = 0\} = \Gamma_2$ (in fact using a system of local coordinates, one can then study the case of every smooth surface Γ_2). We define $x' = (x_1, \dots, x_{N-1})$ and the nonnegative and smooth function ρ_ε through

$$\rho_\varepsilon(x') = \begin{cases} \frac{C}{\varepsilon^{N-1}} \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x'|^2}\right) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases}$$

where

$$C = \left(\int_{B_{N-1}(0,1)} \exp\left(\frac{-1}{1-|\zeta|^2}\right) d\zeta \right)^{-1}.$$

Let $(\omega_{[1/\varepsilon]})_\varepsilon$, where $[1/\varepsilon]$ denotes the entire part of $1/\varepsilon$, be a sequence of open subsets of Γ_2 such that

$$\begin{cases} \omega_1 \subset \omega_2 \subset \dots \subset \omega_{[1/\varepsilon]} \subset \dots \subset \Gamma_2, \\ \bigcup_{\varepsilon} \omega_{[1/\varepsilon]} = \Gamma_2, \\ d(\omega_{[1/\varepsilon]}, \partial\Gamma_2) = \varepsilon. \end{cases}$$

We associate the partition of unity $(\eta_\varepsilon)_\varepsilon$ through

$$\begin{cases} \eta_\varepsilon \in \mathbf{C}_c^\infty(\omega_{[1/\varepsilon]}), \\ \eta_\varepsilon(x') = 1 \text{ in } \omega_{[1/\varepsilon]-1} \text{ (} [1/\varepsilon] - 1 = [1/\varepsilon'], \text{ with } \varepsilon' = \frac{\varepsilon}{1-\varepsilon} \text{)}, \\ 0 \leq \eta_\varepsilon(x') \leq 1, \forall x' \in \Gamma_2, \forall \varepsilon > 0. \end{cases}$$

For $\lambda = (\lambda_1, \dots, \lambda_{N-1}, 0) \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)$, we define the vectorial measure λ^ε through $\lambda^\varepsilon = (\lambda * \rho_\varepsilon) \eta_\varepsilon$. We observe that $\lambda^\varepsilon \in \mathbf{C}_c^\infty(\Gamma_2, \mathbf{R}^N)$ and

$$\begin{cases} \lambda^\varepsilon & \xrightarrow{\varepsilon \rightarrow 0} \lambda & w^*-\mathcal{M}(\Gamma_2, \mathbf{R}^N), \\ |\nabla \lambda^\varepsilon|(x') & \leq \frac{C}{\varepsilon^N} & \forall x' \in \Gamma_2. \end{cases}$$

We build the function w^ε

$$\begin{cases} (w^\varepsilon)_i(x) &= \frac{\varepsilon - x_N}{\varepsilon} (\lambda^\varepsilon)_i(x') & i = 1, \dots, N-1, \forall x \in \Omega, \\ (w^\varepsilon)_N(x) &= \frac{\operatorname{div}(\lambda^\varepsilon(x'))}{2} \left(\frac{(\varepsilon - x_N)^2}{\varepsilon} - \varepsilon \right). \end{cases}$$

We immediately observe that $w^\varepsilon \in \mathbf{H}^1(\Omega, \mathbf{R}^N)$ and

$$\begin{cases} \operatorname{div}(w^\varepsilon) &= 0 & \text{in } \Omega, \\ (w^\varepsilon)_N &= 0 & \text{on } \Gamma_2, \\ w^\varepsilon &= 0 & \text{on } \Gamma_1, \end{cases}$$

that is $w^\varepsilon \in \mathbf{V}_{0,\Gamma_1}(\Omega)$, for every $\varepsilon > 0$. We now define

$$\begin{cases} \varepsilon &= m^{\frac{1}{4N}}, \\ w^m &= w^{\varepsilon^m}, \\ \lambda^m &= \lambda^{\varepsilon^m}. \end{cases}$$

One has

$$\begin{cases} m \int_{\Omega} |\nabla w^m|^2 dx &\leq C\sqrt{m}, \\ J_m(\lambda^m) &= I_m(v_{\lambda^m}) \leq I_m(w^m), \end{cases}$$

hence

$$\limsup_{m \rightarrow 0} J_m(\lambda^m) \leq \limsup_{m \rightarrow 0} I_m(w^m) = J(\lambda).$$

This inequality and (29) end the proof. ■

One has the following result.

Theorem 20 *Let*

$$\begin{aligned} M_i &= \max_{\sigma \in \Gamma_2} \left| \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i(\sigma) \right|, \\ K_i^\pm &= \left\{ \sigma \in \Gamma_2 \mid \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i(\sigma) = \pm M_i \right\}. \end{aligned}$$

We have the following properties.

1. *When m goes to 0, the sequence $\left((u^m/m)|_{\Gamma_2} \right)_m$ converges in the weak* topology of the space $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ to a vectorial measure $\lambda = (\lambda_i)_{i=1, \dots, N}$ such that $\operatorname{supp}(\lambda_i) \subseteq K_i^+ \cup K_i^-$, with λ_i positive on K_i^- and negative on K_i^+ , $i = 1, \dots, N$.*
2. $\int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\lambda_i = -M_i$, $i = 1, \dots, N$.
3. $\lim_{m \rightarrow 0} \int_{\Gamma_2} |u_i^m/m| d\Gamma_2 = |\lambda_i|(\Gamma_2) = M_i$, $i = 1, \dots, N$.

4. When m goes to 0, the sequence $(h_i^m/m)_m$ converges in the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ to a measure $\bar{\lambda}_i$ such that $\text{supp}(\bar{\lambda}_i) \subseteq K_i^+ \cup K_i^-$, $\bar{\lambda}_i$ is positive on K_i^- and negative on K_i^+ , and $|\bar{\lambda}_i|(\Gamma_2) = 1$, $i = 1, \dots, N$.

Proof. One deduces from Proposition 19 and from the properties of the Γ -convergence that $\left((v^m)_{|\Gamma_2}\right)_m = \left((u^m/m)_{|\Gamma_2}\right)_m$ converges in the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$, when m goes to 0, to a measure $\lambda = (\lambda_i)_{i=1, \dots, N}$ such that $J(\lambda) = \min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v)$. Define

$$\mathcal{M}_1(\Gamma_2, \mathbf{R}^N) = \{\mu \in \mathcal{M}(\Gamma_2, \mathbf{R}^N) \mid |\mu_i|(\Gamma_2) = 1, i = 1, \dots, N\}$$

and consider the functional \tilde{J} defined from $[0, +\infty[^N \times \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$ to \mathbf{R} through

$$\begin{aligned} \tilde{J}((t_1, \dots, t_N), (\mu_1, \dots, \mu_N)) &= J((t_1\mu_1, \dots, t_N\mu_N)) \\ &= \frac{1}{2} \sum_{i=1}^N (t_i)^2 + \sum_{i=1}^N t_i \int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i. \end{aligned}$$

One has

$$\min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v) = \min_{\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)} \min_{t_i \geq 0} \tilde{J}((t_1, \dots, t_N), (\mu_1, \dots, \mu_N)). \quad (30)$$

The minimum of (30) with respect to $t = (t_1, \dots, t_N)$ exists if

$$\int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \leq 0, \forall i = 1, \dots, N.$$

Let us now find the minimum with respect to $\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$. One has

$$-\int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \geq -M_i,$$

for every $\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$ such that

$$\int_{\Gamma_2} \left((Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \leq 0, \forall i = 1, \dots, N,$$

the minimum being reached in the case of equality, that is if and only if $\text{supp}(\mu_i) \subset K_i^+ \cup K_i^-$. One has $\lambda_i = M_i\mu_i$, $i = 1, \dots, N$. Remarking that $\bar{\lambda}_i = \mu_i$, one observes that $(h_i^m/m)_m$ converges in the weak* topology of $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$, when m tends to 0, to $\bar{\lambda}_i$, and the same result occurs for the sequence $\left((|u_i^m|_{\Gamma_2}) / \int_{\Gamma_2} |u_i^m| d\Gamma_2\right)_m$. The sequence $(h_i^m/m)_m$ converges in $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ -weak* to a probability measure $\bar{\lambda}_i$ ($\bar{\lambda}_i(\Gamma_2) = 1$) with support in the set of points of Γ_2 where the shear motions, given through $(Id - n \otimes n) \frac{\partial u^0}{\partial n}$, are large for the limit flow described through (27). ■

Remark 21 We thus think that, inside this flow, a thin boundary layer of thinness mh_i occurs in the i -th direction with a probability $\bar{\lambda}_i$ (for every i).

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